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FRICTION FACTORS FOR TURBULENT FLOW IN PIPES

By Edward F. Wilsey

HYDRAULICS DIVISION

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

FRICTION FACTORS FOR TURBULENT FLOW IN PIPES

By Edward F. Wilsey[1]

Synopsis

It was the author's opinion that curves for determining values of the friction factor f could be fitted to the hyperbolic cotangent of x, some function of **R**, the Reynolds number. This is because coth x has an infinite value when x is equal to zero. As x increases, but remains of small magnitude, coth x decreases rapidly. As x increases further in value, the rate of decrease of coth x lessens until, for all practical purposes, coth x reaches unity when x is equal to, or exceeds, 3.5. On this basis two formulas are presented that permit the computation of the friction factor throughout the entire range of Reynolds numbers for turbulent flow in pipes.

Development of the Formulas

Rough Pipes.—The basic formula relating the friction factor f to x some function of the Reynolds number can be written as

$$f = f_{min} \left[\coth\left(\log_{10} x\right)\right]^n \qquad (1)$$

in which n is an undetermined exponent.

J. Nikuradse[2] has shown that the value of the minimum friction factor is

$$f_{min} = \frac{0.25}{\left(\log_{10} 3.7 \frac{D}{k}\right)^2} \qquad (2)$$

in which $\frac{k}{D}$ is the relative roughness of the pipe. Relative roughness is defined as the ratio of the average height k of the protuberances on the pipe surface

Note.—Written comments are invited for publication; the last discussion should be submitted by November 1, 1953.

[1] Former Prof. of Civ. Eng., Ohio Univ., Athens, Ohio. (Mr. Wilsey died on June 30, 1952.)

[2] "Laws of Flow in Rough Pipe," by J. Nikuradse, *Technical Memorandum No. 1292*, National Advisory Committee for Aeronautics, November, 1950, p. 11.

to the diameter D of the pipe. The logarithmic function was introduced to change Reynolds numbers ranging from 4,000 to 10,000,000 to numbers varying from 3.5 to 7.

The first step in determining the final form of Eq. 1 was to find a value for x whose logarithm is approximately 3.5 when the friction factor would be a minimum. The function adopted was

$$x = \mathbf{R} \frac{k}{D} = V \frac{k}{\nu} \quad \text{(3)}$$

in which **R** is the Reynolds number, V is the velocity of the fluid, and ν is the kinematic viscosity.

TABLE 1.—Friction Factors Computed from Eq. 5 and Friction Fa

Reynolds Number $\mathbf{R} \times 10^{-6}$	*D/k*, Inverse Value of the											
	40		70		00		200		400		1,000	
	f_5[a]	f_4[b]	f_5	f_4	f_5	f_4	f_5	f_4	f_5	f_4	f_5	f_4
0.04	5.95	5.9	5.13	5.2	4.82	4.9	4.72	4.7	5.62	4.3	12.1	4.3
0.06	5.78	5.8	4.88	5.0	4.50	4.6	4.18	4.2	4.44	3.9	7.10	3.7
0.08	5.62	5.7	4.76	4.8	4.31	4.4	3.88	3.9	3.86	3.6	5.30	3.4
0.1	5.60	5.7	4.62	4.7	4.22	4.3	3.70	3.7	3.57	3.5	4.45	3.3
0.2	5.47	5.5	4.50	4.5	4.04	4.1	3.38	3.4	3.02	3.1	3.06	2.8
0.4	....	....	....	....	3.93	3.9	3.24	3.3	2.76	2.8	2.51	2.5
0.6	5.35	5.4	4.37	4.3	3.85	3.9	3.18	3.2	2.70	2.7	2.33	2.3
0.8	5.34[c]	5.3	4.35	4.3	3.82	3.8	3.15	3.2	2.65	2.7	2.24	2.2
1.0			4.34	4.3	3.81	3.8	3.13	3.1	2.62	2.6	2.18	2.1
1.4			4.32[c]	4.3	....	....	....	....	....	....	....	....
2.0					3.80[c]	3.8	3.08	3.1	2.56	2.5	2.11	2.1
4.0							3.05[c]	3.0	2.52	2.5	2.04	2.0
6.0									2.50	2.5	2.01	2.0
8.0									2.49[c]	2.5	1.99	2.0
10											1.99	2.0
20											1.97[c]	2.0
40												
60												
80												
100												
200												
400												
600												
800												
f_{min}[d]	5.30		4.29		3.78		3.03		2.48		1.96	

[a] f_5 is herein used to signify the value of the friction factor, multiplied by 100, obtained from Eq. 5. [b] f_4 is formula, Eq. 4. [c] At this point **R**. $k./D$ is equal to 2,000. [d] Values of f_{min} have been computed from Eq. 2.

The final step was to fit the equation to the accepted values of the friction factor by determining a suitable value for n. In order to avoid computation, values of f from the formula of C. F. Colebrook[3]:

$$\frac{1}{\sqrt{f}} = -2 \log_{10} \left(\frac{k}{3.7D} + \frac{2.51}{\mathbf{R} \sqrt{f}} \right) \quad \text{(4)}$$

were taken from curves plotted from this formula.[4] From these values it

[3] "Turbulent Flow in Pipes, with Special Reference to the Transition Region between Smooth and Rough Pipe Laws," by C. F. Colebrook, *Journal*, Inst. of C. E., Vol. 11, 1938-1939, p. 133.

[4] "Hydraulics," by H. W. King, C. O. Wisler, and J. G. Woodburn, John Wiley & Sons, Inc., New York, N. Y., 5th Ed., 1948, p. 192.

was found that $n = 3$, so that:

$$f = f_{min}\left[\coth\left(\log_{10}\mathbf{R}\frac{k}{D}\right)\right]^3 \qquad (5)$$

Table 1 illustrates the fact that values for the friction factor obtained from Eq. 5 approximate the values obtained from Eq. 4 for all values of $\mathbf{R}\frac{k}{D}$ equal to, or greater than, 25. It also illustrates the fact that the friction factor is a minimum when $\mathbf{R}\frac{k}{D}$ is equal to, or exceeds, 2,000. When the value of $\mathbf{R}\frac{k}{D}$ is less than 25, the computed friction factor rises abruptly, indicating a change

CTORS READ FROM CURVES OF THE COLEBROOK FORMULA

Relative Roughness of Pipe 2,000		4,000		10,000		20,000		40,000		Reynolds Number $\mathbf{R} \times 10^{-5}$
f_5	f_4	f_5	f_4	f_5	f_4	f_5	f_4	f_5	f_4	
										0.04
										0.06
										0.08
7.45	3.15									0.1
3.80	2.7	6.41	....							0.2
2.61	2.5	3.27	2.5	7.38	....					0.4
2.30	2.3	2.58	2.2	4.29	2.13					0.6
2.14	2.2	2.24	2.0	3.21	1.94					0.8
2.04	2.0	2.08	1.9	2.70	1.87	4.68	....	17.4	....	1.0
....	....	....	....	....	....	....	....	....	....	1.4
1.86	1.9	1.76	1.8	1.85	1.65	2.38	1.6	4.16	....	2.0
1.79	1.8	1.60	1.6	1.52	1.5	1.64	1.4	2.12	....	4.0
1.75	1.8	1.57	1.6	1.41	1.4	1.45	1.4	1.67	1.35	6.0
1.74	1.7	1.54	1.5	1.36	1.4	1.34	1.3	1.45	1.3	8.0
1.72	1.7	1.52	1.5	1.33	1.3	1.26	1.2	1.34	1.2	10
1.70	1.7	1.49	1.5	1.27	1.3	1.17	1.2	1.14	1.1	20
1.68[c]	1.7	1.46	1.5	1.24	1.2	1.12	1.1	1.04	1.0	40
		1.45	1.5	1.22	1.2	1.10	1.1	1.02	1.0	60
		1.45[c]	1.5	1.21	1.2	1.09	1.1	1.00	1.0	80
				1.20	1.2	1.07	1.1	0.98	0.99	100
				1.20[c]	1.2	1.06	1.1	0.96	0.97	200
						1.06[c]	1.1	0.95	0.96	400
								0.94	0.95	600
								0.94[c]	0.94	800
1.67		1.44		1.19		1.05		0.933		f_{min}[d]

herein used to signify the value of the friction factor, multiplied by 100, obtained from curves of the Colebrook

in the type of flow. When $\mathbf{R}\frac{k}{D}$ is approximately equal to 25, smooth-pipe flow is similar to rough-pipe flow.

Smooth Pipes.—For smooth pipes $\frac{k}{D}$ is equal to zero, and Eq. 5 must be modified. If the position of $\log_{10} x$ on the hyperbolic cotangent curve were moved, the exponent in the formula for the friction factor would change. Further investigation resulted in

$$f = 0.001\left[\coth\left(\frac{\log_{10}\mathbf{R}}{12.8}\right)\right]^3 \qquad (6)$$

The coefficient 12.8 was chosen in order that the smooth-pipe equation and the rough-pipe equation would be equal when $\mathbf{R}\,\frac{k}{D}$ is equal to 25. Eq. 6 illustrates that the minimum friction factor for a smooth pipe is 0.001.

The values for the friction factor in smooth pipes agree favorably with the values computed from the formula developed by L. Prandtl, T. von Kármán, Hon. M. ASCE, and Mr. Nikuradse,[5,6]

$$\frac{1}{\sqrt{f}} = 2 \log_{10} \frac{\mathbf{R}\sqrt{f}}{2.51} \qquad (7)$$

Other formulas for the friction factor of smooth pipes have been developed. Mr. Nikuradse has suggested the expression[7]

$$f = 0.0032 + 0.221\,\mathbf{R}^{-0.237} \qquad (8)$$

and T. B. Drew, E. C. Koo, and W. H. McAdams proposed the following[8]:

$$f = 0.0056 + 0.500\,\mathbf{R}^{-0.32} \qquad (9)$$

TABLE 2.—COMPARISON OF SMOOTH PIPE FRICTION FACTORS

Reynolds number $\mathbf{R} \times 10^{-5}$	Wilsey (Eq. 6)	Prandtl, von Kármán, and Nikuradse (Eq. 7)	Nikuradse (Eq. 8)	Drew, Koo, and McAdams (Eq. 9)
0.04	0.0482	0.040	0.034	0.041
0.06	0.0421	0.035	0.031	0.036
0.08	0.0386	0.033	0.029	0.034
0.10	0.0328	0.031	0.028	0.032
0.20	0.0292	0.026	0.024	0.027
0.40	0.0244	0.022	0.021	0.022
0.60	0.0217	0.020	0.020	0.020
0.80	0.0203	0.019	0.018	0.019
1.00	0.0194	0.018	0.018	0.019
2.00	0.0166	0.015	0.015	0.016
4.00	0.0142	0.014	0.014	0.013
6.00	0.0131	0.013	0.013	0.013
8.00	0.0125	0.012	0.012	0.012
10.00	0.0120	0.012	0.012	0.011
20.00	0.010	0.010	0.010	0.010
40.00	0.0093	0.0093	0.0092	0.0095
60.00	0.0087	0.0087	0.0087	0.0090
80.00	0.0084	0.0083	0.0083	0.0087
100.00	0.0081	0.0080	0.0081	0.0085

Table 2 shows a comparison of values for the friction factors obtained from Eqs. 6, 7, 8, and 9 for Reynolds numbers ranging in value from 4,000 to 10,000,000. The friction factors from Eq. 7 were read from curves plotted by R. C. Binder.[9] There is a fair agreement for values of the friction factors obtained with Reynolds numbers greater than 100,000. For values of $\mathbf{R}$ less than 100,000, they diverge because in this region coth x changes quite rapidly, and laboratory measurements are subject to error. Any other divergency can

[5] "Turbulence and Skin Friction," by T. von Kármán, *Journal of the Aeronautical Sciences*, January 1934, p. 1.

[6] "Mechanische Ahnlichkeit und Turbulenz," by T. von Kármán, *Proceedings*, Third Congress for Applied Mech., Stockholm, Sweden, Vol. 1, 1930, p. 85.

[7] "Gesetzmässigkeit der turbulenten Strömung in glatten Röhren," by J. Nikuradse, *Verein Deutscher Ingenieur, Forschungsscheft No. 356*, 1932.

[8] "The Friction Factor for Clean Round Pipes," by T. B. Drew, E. C. Koo, and W. H. McAdams, *Transactions*, Am. Inst. of Chemical Engrs., Vol. 28, 1932, p. 56.

[9] "Fluid Mechanics," by R. C. Binder, Prentice-Hall, New York, N. Y., 2d Ed., 1949, p. 85.

be explained by the realization that Eqs. 7, 8, and 9 were fitted to data, whereas Eq. 6 was designed to meet Eq. 5 at the point where $\mathbf{R}\,\frac{k}{D}$ is equal to 25.

In order to investigate further the validity of Eq. 6, a comparison was made with the data of measured friction factors for smooth brass pipe of Victor L. Streeter,[10] M. ASCE. Table 3 shows these data arranged in the order

TABLE 3.—COMPARISON OF FRICTION FACTORS FOR SMOOTH BRASS PIPES

Reynolds number $\mathbf{R} \times 10^{-5}$	Streeter	Wilsey	Prandtl, von Kármán, and Nikuradse	Reynolds number $\mathbf{R} \times 10^{-5}$	Streeter	Wilsey	Prandtl, von Kármán, and Nikuradse
0.179	0.0310	0.0300	0.026	5.35	0.0129	0.0135	0.0129
0.238	0.0201	0.0259	0.0248	5.42	0.0136	0.0135	0.0129
0.400	0.0198	0.0244	0.0219	5.47	0.0134	0.0135	0.0128
0.407	0.0194	0.0241	0.0218	5.50	0.0131	0.0134	0.0128
0.482	0.0206	0.0229	0.0208	5.61	0.0130	0.0134	0.0128
0.575	0.0216	0.0224	0.0200	5.92	0.0133	0.0135	0.0127
0.643	0.0188	0.0215	0.0195	5.96	0.0129	0.0133	0.0127
0.736	0.0198	0.0208	0.0190	6.08	0.0126	0.0131	0.0127
0.752	0.0191	0.0206	0.0190	6.19	0.0129	0.0131	0.0126
0.817	0.0196	0.0203	0.0185	6.27	0.0126	0.0130	0.0125
0.860	0.0182	0.0201	0.0180	6.34	0.0130	0.0130	0.0125
0.933	0.0182	0.0197	0.0180	6.49	0.0124	0.0129	0.0125
1.034	0.0177	0.0192	0.0176	6.61	0.0130	0.0129	0.0123
1.147	0.0174	0.0186	0.0175	6.76	0.0129	0.0128	0.0123
1.33	0.0168	0.0182	0.0170	6.82	0.0126	0.0128	0.0123
1.51	0.0168	0.0176	0.0165	6.85	0.0126	0.0127	0.0122
1.63	0.0164	0.0174	0.0160	6.86	0.0126	0.0127	0.0122
1.68	0.0165	0.0172	0.0160	7.21	0.0124	0.0126	0.0121
1.88	0.0160	0.0170	0.0157	7.28	0.0127	0.0126	0.0121
2.10	0.0158	0.0166	0.0153	7.67	0.0126	0.0126	0.0120
2.21	0.0125	0.0164	0.0153	7.76	0.0123	0.0125	0.0120
2.25	0.0160	0.0160	0.0150	7.80	0.0122	0.0125	0.0120
2.44	0.0141	0.0158	0.0148	8.05	0.0124	0.0124	0.0120
2.61	0.0143	0.0154	0.0148	8.15	0.0125	0.0123	0.0120
2.73	0.0149	0.0153	0.0147	8.36	0.0120	0.0123	0.0119
2.73	0.0156	0.0153	0.0147	8.40	0.0120	0.0122	0.0119
2.75	0.0151	0.0153	0.0147	8.71	0.0119	0.0122	0.0119
2.84	0.0147	0.0153	0.0145	8.73	0.0120	0.0122	0.0119
3.16	0.0144	0.0151	0.0141	9.42	0.0121	0.0121	0.0118
3.16	0.0143	0.0151	0.0141	9.46	0.0121	0.0120	0.0118
3.33	0.0158	0.0149	0.0140	9.46	0.0118	0.0120	0.0118
3.40	0.0143	0.0149	0.0140	9.48	0.0119	0.0119	0.0118
3.50	0.0141	0.0147	0.0140	10.19	0.0117	0.0118	0.0117
3.52	0.0141	0.0147	0.0139	10.62	0.0115	0.0118	0.0117
3.73	0.0137	0.0145	0.0138	10.85	0.0113	0.0117	0.0116
3.73	0.0142	0.0145	0.0138	11.02	0.0114	0.0117	0.0116
3.93	0.0129	0.0144	0.0138	11.08	0.0114	0.0117	0.0116
3.96	0.0140	0.0143	0.0138	11.22	0.0114	0.0116	0.0115
4.09	0.0132	0.0142	0.0135	11.53	0.0108	0.0116	0.0115
4.26	0.0139	0.0141	0.0135	12.02	0.0111	0.0116	0.0114
4.48	0.0139	0.0140	0.0133	12.30	0.0107	0.0115	0.0113
4.66	0.0140	0.0139	0.0132	12.39	0.0111	0.0115	0.0113
4.67	0.0140	0.0139	0.0131	13.30	0.0110	0.0114	0.0112
4.78	0.0132	0.0138	0.0131	13.93	0.0107	0.0112	0.0112
4.90	0.0136	0.0138	0.0131	14.42	0.0108	0.0112	0.0110
5.06	0.0137	0.0137	0.0130	15.00	0.0104	0.0111	0.0109
5.17	0.0127	0.0137	0.0130	15.62	0.0106	0.0109	0.0108
5.22	0.0127	0.0136	0.0130	15.96	0.0108	0.0106	0.0107

of increasing Reynolds numbers. The measured friction factors are compared with values from Eq. 6 and values obtained from a curve of Mr. von Kármán's equation.[9] There is a scattering of the measured friction factors for the lower Reynolds numbers. For these low values of Reynolds numbers there appears to be no way of determining the true values of f. As the Reynolds number increases, Eq. 5 conforms more closely with the experimental results.

[10] "Frictional Resistance in Artificially Roughened Pipes," by Victor L. Streeter, *Transactions*, ASCE, Vol. 101, 1936, p. 687.

Conclusions

To facilitate the use of the equations which have been developed for determining a suitable value of the friction factor, the following procedure outline is presented:

1. If $\mathbf{R}\frac{k}{D}$ is greater than 25 and less than 2,000, use Eq. 5.
2. When $\mathbf{R}\frac{k}{D}$ is greater than 2,000, use Eq. 2.
3. Eq. 6 should be used when $\mathbf{R}\frac{k}{D}$ is less than 25.

When $\mathbf{R}$ is less than 2,000—that is, when the flow is laminar—the familiar equation, $f = 64/\mathbf{R}$, should be used.

Eqs. 5 and 2 give results that agree closely with accepted values for the friction factor. The results from Eq. 6 diverge somewhat from the results obtained with other formulas when the Reynolds number has a low value. This divergency indicates a need for further investigation of flow having low values of the Reynolds number.

It may be felt that the new formulas are complicated. This is not so, since all that is needed for the computations is a set of mathematical tables. In defense of simple formulas, the following has been stated[11]:

> "Daniel Bernoulli often said to me to eschew all complicated formulas; he believing that the organization of Nature is too simple to lead to them; and should one find such, the explanation is that one's computations were based upon false hypotheses."

However, the writer feels that the findings in scientific research are truly a revelation, and the way to reveal "Nature's" plan is to work hard, pray to God for guidance, and be sceptical of authoritarianism.

[11] "Biographies Relating to the History of Civilization in Switzerland," by R. Wolf, Vol. III, 1860, p. 176.